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LOW-FREQUENCY PROPAGATION IN AN EXPONENTIAL IONOSPHERIC LAYER

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Abstract

Recently, Heading and Whipple treated the problem of propagation of long radio waves in an exponential ionospheric layer. They dealt with the case of an obliquely incident plane wave, a vertical magnetic field, and low collision frequency. The case treated here is that of vertical incidence, oblique magnetic field, and both low and high collision frequency. The results obtained do not agree with experimental data. This is probably due to the fact that coupling effects were neglected. An example is given in the Appendix which shows that the omission of coupling terms, however small their coefficients may be, is not always justified.

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1. Introduction

Recently Heading and Whipple[1] treated the problem of propagation of long radio waves in an exponential ionospheric layer. They dealt with the case of an obliquely incident plane wave and a vertical magnetic field, and assumed both wave frequency and collision frequency to be small compared to gyrofrequency. The validity of the latter assumption is open to question. Nevertheless the computed results* are reasonable. The overall reflection coefficient is found to increase with angle of incidence, which agrees with experimental data.

Because of the success of Heading and Whipple in solving the problem under the assumptions mentioned, it was deemed worthwhile to investigate the case of vertical incidence and an oblique magnetic field. In this case it was necessary to use somewhat more restrictive assumptions and some additional approximations in order to calculate the reflections coefficients. Unfortunately, the results so obtained do not agree well with experimental data. In view of this fact we may wonder just what assumption or approximation is responsible for the discrepancy. Since the approximations used in this paper and in that of Heading and Whipple are very similar, the similarity between the results of both papers will be pointed out whenever possible.

2. Oblique Reflection in Vertical Magnetic Field Collision Frequency Small

We shall reproduce here, in bare outline, Heading and Whipple's theory of oblique reflection in vertical magnetic field. The differential equations for the components of the electric field for this case were derived by Wilkes[2]. They are

$$(1)\quad \frac{d^2}{dz^2}(E_x \pm iE_y) + \frac{d}{dz}\left(m\frac{d}{dz}E_x\right) + k^2\left(1 - \frac{X}{1+iZ\pm Y}\right)(E_x \pm iE_y) \mp ik^2\sin^2\theta E_y = 0$$

$$(2)\quad E_z = (m/ik\sin\theta)dE_x/dz$$

$$\text{where}\qquad m = \frac{\sin^2\theta}{\cos^2\theta - \frac{X}{1+iZ}}\ ,$$

---

* Results were computed for 16 Kc and 80 kc waves with collisional frequency $\nu = 5 \times 10^5\ \text{sec.}^{-1}$; the electron density was assumed to have Chapman distribution with a penetration frequency of 1.27 mc and a scale height of 6km.

$X = \omega_o^2/\omega^2$,

$Y = \omega_H/\omega$,

$Z = \nu/\omega$,

$k = \omega c = 2\pi/\lambda$

ω - wave frequency,

ω_o - critical (plasma) frequency

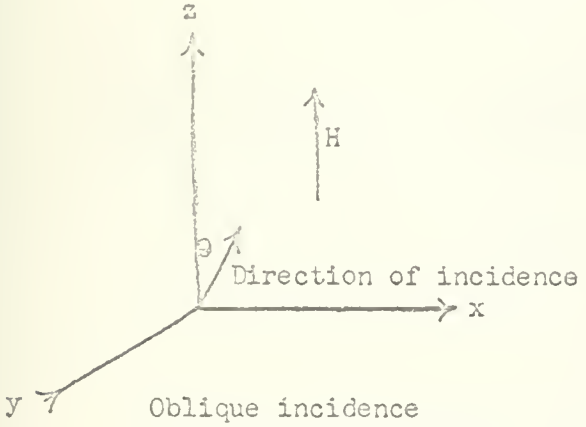
ω_H - gyrofrequency,

ν - collision frequency,

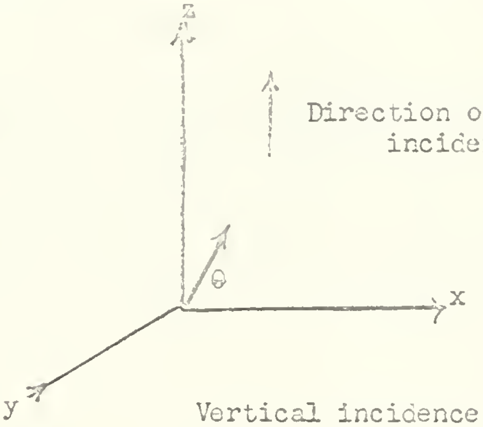
Θ - angle between direction of incidence (in the xz-plane) and the z axis,

and the time dependence was taken to be $e^{-i\omega t}$.

If Y is much greater than both Z and unity, and X increases with height, we can simplify the problem by discussing it separately in two regions. Region I, the lower one, is centered at the height given by $X(z) = 1$. In this region X is small compared to Y, thus making the term $X/(1+iZ\mp Y)$ small compared to unity. In Region II, the upper one, centered at the height given by $X(z) = Y$,



Oblique incidence
Vertical magnetic field



Vertical incidence
Oblique magnetic field

X is much greater than $|1+iZ|$, and therefore m is small.

In Region I, then, if we drop the term $X/(1+iZ\mp Y)$, equations (1) reduce to two separate equations for E_x and E_y. Since it is more convenient to deal with E_z than with E_x we can make use of equation (2) and obtain the following set of equations:

$$\frac{d^2}{dz^2}E_y + k^2\cos^2\theta E_y = 0 \tag{3}$$

$$\frac{d^2}{dz^2}\left(\frac{1+m}{m}E_z\right) + \frac{k^2}{m}E_z = 0. \tag{4}$$

The latter may be written in the form

$$\text{(5)}\quad \phi_{zz} + \frac{k^2}{1+m}\phi = 0\ , \qquad \phi = \frac{1+m}{m}E_z\ .$$

We see then that in this region horizontally and vertically* polarized waves are propagated independently. The horizontally polarized wave is unaffected by the medium, as is evident from equation (3). If we substitute the definition of m into (5), we obtain

$$\phi_{zz} + k^2\,\frac{(1+iZ)\cos^2\theta - X}{(1+iZ) - X}\,\phi = 0. \tag{6}$$

We have here a wave equation for ϕ, with the refractive index changing from $\cos\theta$ to unity as X increases from zero to infinity. For small values of Z the refractive index has typical resonance behavior near $X = 1$.

If X increases exponentially with height, equation (6) is but a special case of a wave equation treated in detail by Epstein[3]; if we let $\varepsilon_3 = 0$ in the equation he considered it becomes

$$\phi_{zz} + k^2\,\frac{\varepsilon_1 + \varepsilon_2 e^{\zeta}}{1 + e^{\zeta}}\,\phi = 0\ , \qquad \zeta = C(z - z_o)\ . \tag{7}$$

This equation can be solved in terms of hypergeometric functions. The reflection and transmission coefficients can be obtained in terms of gamma functions of the parameters of the problem. Whatever the variation of X with height, the vertically polarized component is partially reflected at this level.

In Region II we may neglect the second term in equation (1), thus obtaining

$$\frac{d^2}{dz^2}E_x + k^2E_x = ik\,\frac{\lambda}{Y}\,E_y \tag{8}$$

---

* Strictly speaking there is no vertically polarized wave; we mean here the wave polarized in the plane of incidence.

$$\frac{d^2}{dz^2} E_y + k^2 \cos^2 \Theta E_y = -ik \frac{X}{Y} E_x \,. \tag{9}$$

These are coupled wave equations; they cannot be reduced to a system of independent second-order equations. They can be solved, however, in terms of generalized hypergeometric functions, if X/Y varies exponentially with height. Four independent solutions can be found. Two of these correspond to upgoing incident waves, and the other two to downcoming waves incident on the ionosphere from the top. From the asymptotic properties of the former two solutions we obtain the reflection and coupling coefficients for Region II.

3. Vertical Propagation in Oblique Magnetic Field; Collision Frequency Small

Rydbeck[4] has shown that the differential equations for the case of vertical incidence and oblique magnetic field are

$$\frac{d^2}{dz^2} \Pi_\pm + \left[k^2 \varepsilon_\pm - \frac{1}{4} \left(\frac{d\delta/dz}{1+\delta^2} \right) \right] \Pi_\pm \tag{10}$$

$$= -\frac{i}{2} \Pi_\mp \frac{d}{dz} \left(\frac{d\delta/dz}{1+\delta^2} \right) - i \frac{d\Pi_\mp}{dz} \cdot \frac{d\delta/dz}{1+\delta^2} \,,$$

where the two dependent variables Π_+ and Π_- replace the components E_x and E_y of the electric field which are used directly by Wilkes. Here the total field is first resolved into components having polarization (E_y/E_x) equal u and 1/u :

$$E_x^+ = (1-u^2)^{-1}(E_x - uE_y) \,, \qquad E_y^+ = uE_x^+ \,, \tag{11}$$

$$E_y^- = (1-u^2)^{-1}(E_y - uE_x) \,, \qquad E_x^- = uE_y^- \,,$$

where

$$u = i \left[\delta - (\delta^2 + 1)^{1/2} \right] \tag{12}$$

$$\delta = \frac{1}{2} \frac{\sin^2 \Theta}{\cos \Theta} \quad \frac{Y}{X - 1 - iZ} \,, \tag{13}$$

or, in terms of the critical collision frequency $\nu_c = \omega_H \sin^2\Theta / 2\cos\Theta$,

$$\delta = \frac{Z_c}{X - 1 - iZ} \,, \tag{14}$$

where $$Z_c = \nu_c / \omega \,.$$

Then we define

$$\Pi^{+} = (1-u^2)^{1/2}E_x^+ \tag{15}$$

$$\Pi^{-} = (1-u^2)^{1/2}E_y^- \ .$$

3.1 Coupling Terms

We see that the coefficients of the coupling terms in equation (10) are $(d\delta/dz)/(1+\delta^2)$ and its derivative. We have

$$\frac{d\delta/dz}{1+\delta^2} = \frac{-Z_c}{Z_c^2 + (X-1-iZ)^2}\ \frac{dX}{dz} \ . \tag{16}$$

We see that this quantity can become very large if $Z \approx Z_c$. Instead of assuming that $\nu \ll \omega_H$, as Whipple does, let us assume here that $\nu \ll \nu_c$. This coupling term is then negligible if

$$dX/dz \ll kZ_c \ . \tag{17}$$

Since Z_c is large, this condition is easily satisfied. In the remainder of this paper coupling terms will be neglected. This may not be justified, even though the coupling coefficients are small. It may be that the limit of solutions of the equations with smaller and smaller coupling coefficients differs from the solution of the equations with coupling terms omitted. An example of such a discrepancy is discussed in the Appendix.

3.2 Limiting Polarization

Since we are interested only in the asymptotic properties of wave functions well below the reflecting regions of the ionosphere, we need to consider only the limiting polarization (as X -> 0) of the characteristic waves used here. From equation (14) it is evident that δ is large when X = 0. Hence u is given approximately by

$$u = \frac{-i}{2\delta} = \frac{1+iZ}{-2iZ_c} \ . \tag{18}$$

Since $|u| \ll 1$, we are dealing here with two approximately linearly polariz waves.

3.3 Dielectric Constants

We shall now investigate the dielectric constants, $\varepsilon_{\pm}$, in equation (10). These are given by the Appleton-Hartree formula [5]

(19) $$\varepsilon_{\pm} = 1 - \frac{X}{1+iZ + \dfrac{Y^2\sin^2\theta}{2(X-1-iZ)} \pm \dfrac{1}{2}\left[\dfrac{Y^4\sin^4\theta}{(X-1-iZ)^2} + 4Y^2\cos^2\theta\right]^{1/2}} .$$

In the lower region (Region I of Heading and Whipple), where $X \sim 1$ and therefore $X \ll Y$, we obtain the following approximations:

(i) For the ordinary (+) ray we can neglect the $4Y^2\cos^2\theta$ term under the radical. The $(1+iZ)$ term is negligible compared to $Y^2\sin^2\theta/(x-1-iZ)$. Hence, we obtain

(20) $$\varepsilon_{+} \approx 1 - \frac{X(X-1-iZ)}{Y^2\sin^2\theta} .$$

(ii) For the extraordinary (-) ray the $4Y^2\cos^2\theta$ term cannot be neglected because of the near cancellation of the radical and the preceding term. Retaining two terms in the binomial expansion of the radical, we get

(21) $$\varepsilon_{-} \approx 1 - \frac{X\sin^2\theta}{1+iZ-X\cos^2\theta} .$$

Figure 1 shows a comparison between values of the dielectric constant obtained from Eq. (21) (circles) and those computed from the Appleton-Hartree formula by Kelso.[6] Thus we see that the (+) component propagates almost as if the ionosphere were not there, unless $\sin\theta$ is very small. The other component, however, behaves quite differently: the dielectric constant shows a resonance, and it changes from unity at $X = 0$ to $\sec^2\theta$ for large X. The differential equation satisfied by this component is

(22) $$\frac{d^2}{dz^2}\Pi_{-} + k^2\sec^2\theta\,\frac{(1+iZ)\cos^2\theta - X\cos^2\theta}{(1+iZ) - X\cos^2\theta}\,\Pi_{-} = 0.$$

The similarity between this equation and equation (6) is quite evident. In the case of exponential layer

(23) $$X = \exp\frac{x-x_0}{h} ,$$

we can follow Epstein's method of solving the equation.

Let

(24) $$u = \frac{-X\cos^2\theta}{1+iZ} = e^{\xi+i\eta} ;$$

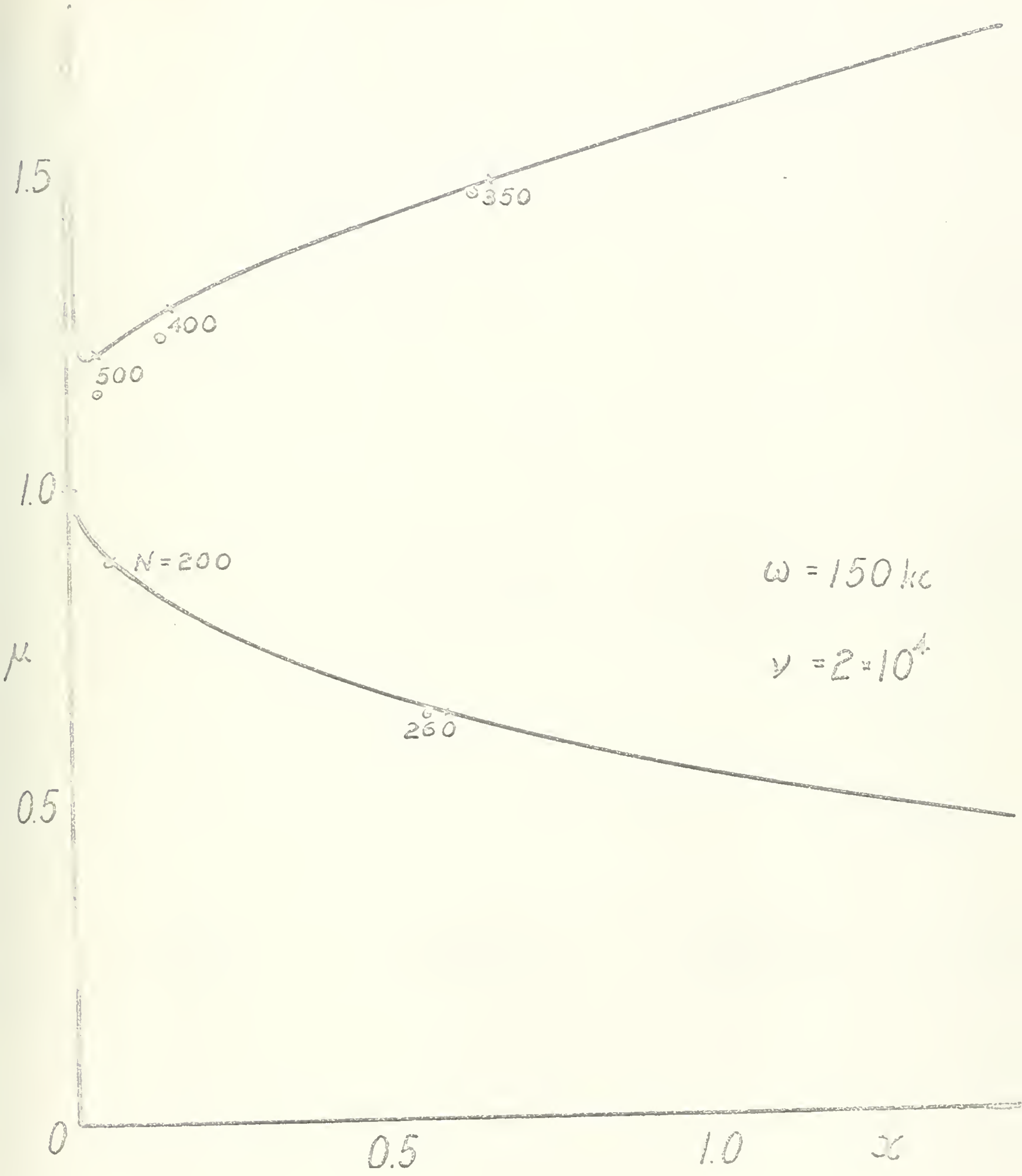


Figure 1

Complex Dielectric Constant for the
Extraordinary Ray in Region I

—— Approximate Curve

⊙ Exact values (from Kelso)

$$\zeta = (x-x_o)/h$$

$$\vartheta = \pi - \tan^{-1}z.$$

Equation (22) becomes then

$$(25) \quad \Pi_{uu} + u^{-1}\,\Pi_u + k^2h^2\left[u^{-2} + (\sec^2\theta - 1)(1+u)^{-1}u^{-1}\right]\Pi = 0\ ;$$

(25) and Epstein's eq. (12) are identical if in the latter $c_1 = 1$, $c_2 = 0$, $p = 0$. Now we put

$$(26) \qquad \Pi = u^{ikh}f\ ,$$

and then eq. (25) becomes

$$(27) \quad u(u+1)f'' + (2ikh+1)(u+1)f' + (ikh+ikh\sec\theta)(ikh-ikh\sec\theta)f = 0,$$

or the hypergeometric equation. The solutions of (25) are then

$$(28a) \qquad \Pi_1^{\pm} = u^{\pm ikh}\,F(\pm ikh+ikh\sec\theta,\ \pm ikh-ikh\sec\theta;\ \pm 2ikh+1;\ -u),$$

or, in descending powers of u,

$$(28b) \qquad \Pi_2^{\pm} = u^{\mp ikh\sec\theta}\,F(ikh\pm ikh\sec\theta,\ -ikh\pm ikh\sec\theta;\pm 2ikh\sec\theta+1;-u^{-1}).$$

We note that for $x \to \infty$, or $|u| \to \infty$, the Π_2^- solution is appropriate, since it corresponds to an outgoing wave (with the time dependence $\exp(-i\omega t)$, as used throughout the paper). For $x \to -\infty$, or $u \to o$, the Π_1^+ and Π_1^- functions represent incoming and outgoing waves respectively. Their limiting behavior is

$$(29) \qquad \Pi_1^{\pm} \sim u^{\pm ikh} = (e^{\zeta+i\vartheta})^{\pm ikh} = e^{\pm ik(x-x_o)}\ e^{\mp kh\vartheta}.$$

In order to calculate the reflection coefficient, we make use of the circuit relation

$$(30) \qquad \Pi_2^- = A(ikh-ikh\sec\theta,-ikh-ikh\sec\theta)\quad \Pi_1^+$$
$$+ A(-ikh-ikh\sec\theta,\ ikh-ikh\sec\theta)\ \Pi_1^-\ ,$$

where

$$A(\alpha,\beta) \quad = \frac{\Gamma(\alpha+\beta+1)\ \Gamma(\beta-\alpha)}{\Gamma(\alpha)\ \Gamma(\beta)}\ .$$

We have here an expression relating the solution of the problem which satisfies the outgoing wave condition at infinity to two functions which behave like incident and reflected wave at $(-\infty)$. Combining (29) and (30), we obtain

the reflection coefficient as the following ratio of the complex amplitudes of the incident and reflected parts of the total wave solution:

$$(31)\qquad R = \frac{\Gamma(2ikh)\,\Gamma(-ikh-ikh\sec\theta)\,\Gamma(-ikh-ikh\sec\theta+1)}{\Gamma(-2ikh)\,\Gamma(ikh-ikh\sec\theta)\,\Gamma(ikh-ikh\sec\theta+1)}\,e^{2kh\phi}$$

or

$$(32)\qquad R = \frac{1+\cos\theta}{1-\cos\theta}\,\frac{\Gamma(2ikh)}{\Gamma(-2ikh)}\left[\frac{\Gamma(ikh-ikh\sec\theta)}{\Gamma(-ikh-ikh\sec\theta)}\right]^{-2} e^{2kh(\pi-\tan^{-1}Z)}.$$

Using the relation

$$\left|\Gamma(ix)\right|^2 = \Gamma(ix)\,\Gamma(-ix) = \pi/x\,\sinh\pi x$$

we can calculate the absolute value of the reflection coefficient:

$$(33)\qquad \left|R\right| = \frac{\sinh\pi kh(\sec\theta+1)}{\sinh\pi kh(\sec\theta-1)}\,\exp\left[2kh\,(\pi-\tan^{-1}Z)\right]$$

$$= \frac{1-\exp\left[2\pi kh(1-\sec\theta)\right]}{1-\exp\left[2\pi kh(-1-\sec\theta)\right]}\,\exp(-2\pi kh\,\tan^{-1}Z).$$

Since Z is a function of frequency, define Z_o as the value of Z when kh = 1. Then

$$(34)\qquad Z = \nu/\omega = \nu ch/kh = Z_o/kh\,.$$

Equations (33) and (34) give the relation between the frequency and the magnitude of the reflection coefficient for the extraordinary wave in Region I. A graph of $|R|$ for some values of θ and Z_o is shown on Fig. 2.

In the upper region, neglecting (1+iZ) compared to X or Y, we get

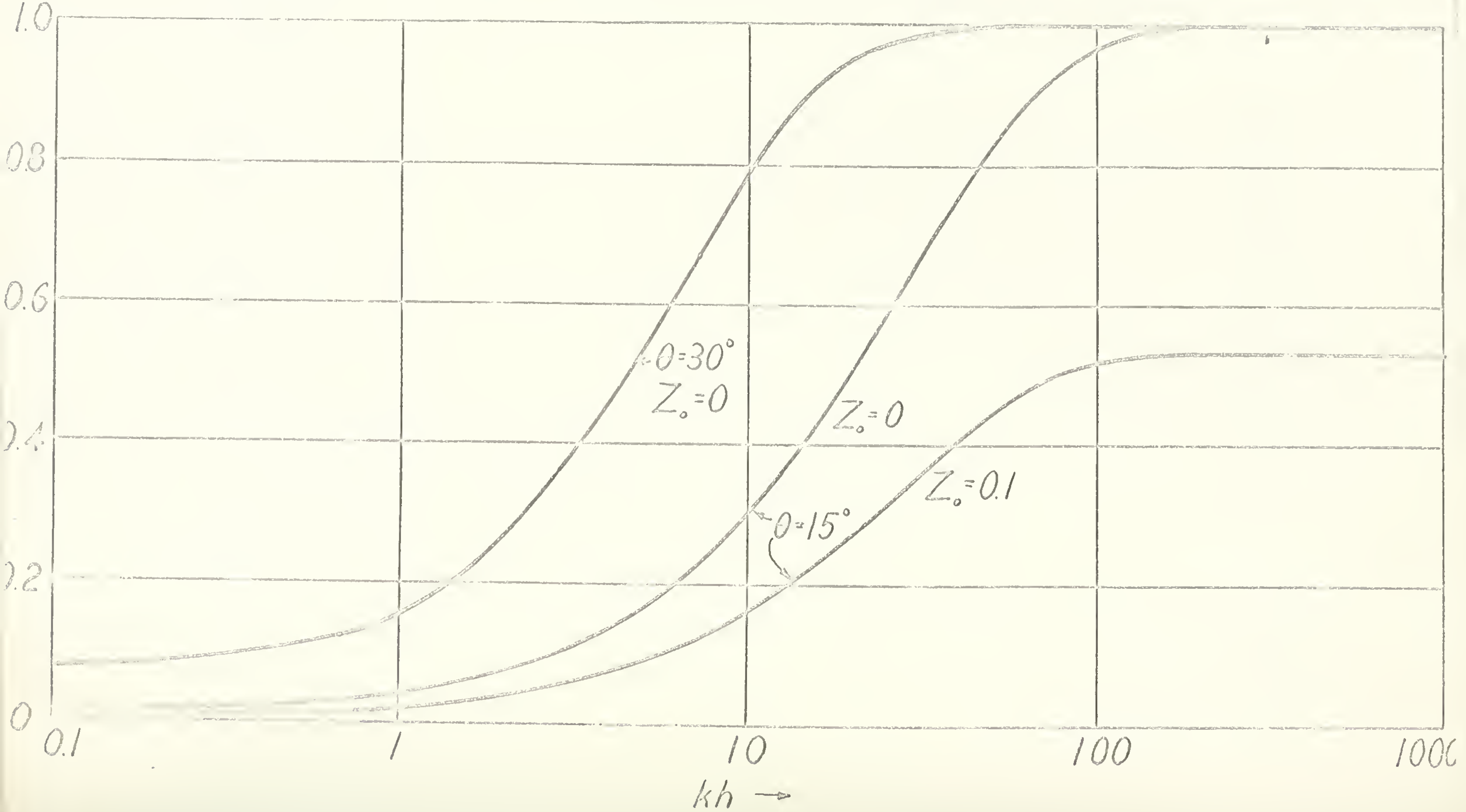
$$(35)\qquad \varepsilon_{\pm} \approx 1 - \left[\frac{Y^2\sin^2\theta}{2X^2} \pm \frac{Y}{2X}\left(4\cos^2\theta + \frac{Y^2}{X^2}\sin^4\theta\right)^{1/2}\right]^{-1}.$$

Evidently the dielectric constants are functions of a single parameter

$$(36)\qquad r = X/Y = \omega_o^2/\omega\omega_H\,.$$

The dielectric constants are real and independent of collision frequency, just as in the case treated by Whipple. Equation (35) can be simplified to

REFLECTION COEFFICIENT IN REGION I



(37) $$\varepsilon_{\pm} \approx (1+\frac{1}{2}\tan^2\theta) \mp (r^2\sec^2\theta + \frac{1}{4}\tan^4\theta)^{1/2}.$$

From this we see that ε_-, the dielectric constant which exhibits a resonance in region I, is positive in Region II. The other dielectric constant, ε_+, has a zero at r=1. If we had used no approximations this zero would be found at $r = 1 + Y^{-1}$; the shift to $r = 1$ is due to our having neglected terms of the order Y^{-1} compared to unity. Figure 3 is a graph of ε_+ and ε_- for $\theta = 30^0$. For large r we can approximate ε by

(38) $$\varepsilon_{\pm} = 1 + \frac{1}{2}\tan^2\theta \mp r\sec\theta ,$$

while for small values of r

(39) $$\varepsilon_+ \approx 1$$
$$\varepsilon_- \approx \sec^2\theta.$$

The latter values are identical with those for large X in Region I.

In Region I the case of exponential variation of X with height can be solved exactly. In Region II the problem can be solved approximately by replacing (37) by

(40) $$\varepsilon_{\pm} \approx (1+\frac{1}{2}\tan^2\theta) \mp (\frac{1}{2}\tan^2\theta + r\sec\theta).$$

This approximation has the advantage of being good for any θ for both small and large values of r. For intermediate values of r it is good for small θ. For larger values of θ a different approximation might have to be used.

In Region II, then, using the approximation (40) and exponential variation (23), the dielectric constant is a sum of a constant and an exponentially varying term for both waves. We can consider in general the differential equation

(41) $$y_{xx} + \kappa^2 \varepsilon y = 0$$

where

$$\varepsilon = 1 + \exp\left[i\alpha + (x-x_0)/h \right].$$

In this case κ is not the free space wave number, but the limiting value of the wave number as $x \to -\infty$. If we now employ the change of variable

(42) $$\zeta = \exp(x/2h) ,$$

equation (41) transforms into

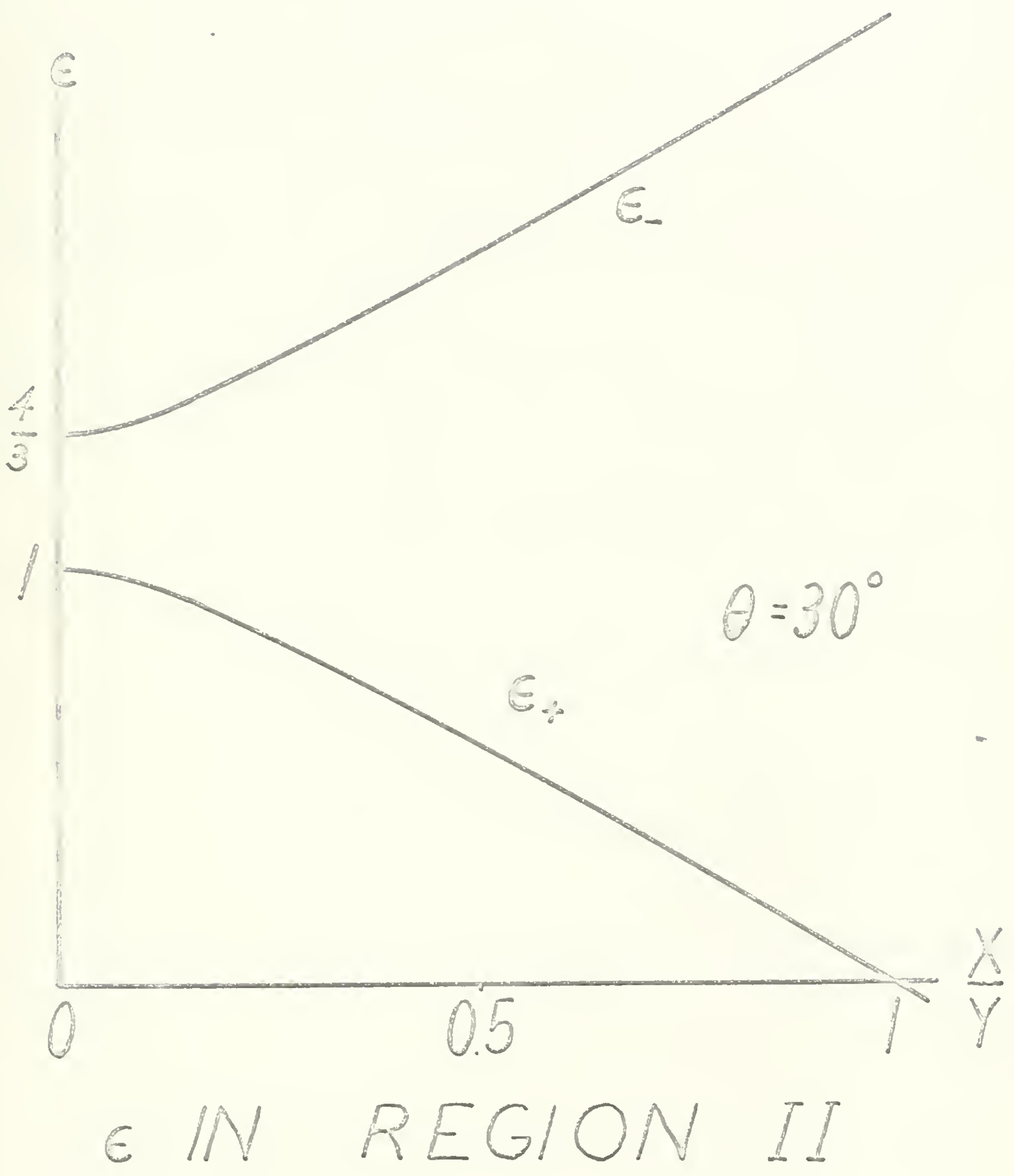


Figure 3

$$(43) \qquad y_{\zeta\zeta} + \frac{1}{\zeta}\, y_{\zeta} + \left[\frac{4h^2\kappa^2}{\zeta^2} + 4h^2 \exp\left(i\alpha - \frac{x_o}{h}\right) \right] y = 0.$$

This equation has as solutions Bessel functions of order $n = \pm 2i\kappa h$ and argument $2h\zeta \exp \frac{1}{2}\left(i\alpha - \frac{x_o}{h}\right)$. Since we require the solution to behave like an outgoing wave at infinity,

$$(44) \qquad y = H^{(1)}_{2i\kappa h} \left[2h \exp \frac{1}{2}\left(i\alpha + \frac{x-x_o}{h}\right) \right].$$

In order to investigate the behavior of the solution at $(-\infty)$ we express it in terms of $J_{2i\kappa h}$ and $J_{-2i\kappa h}$:

$$(45) \qquad H^{(1)}_n = \frac{i}{\sin n\pi}\left(e^{-in\pi} J_n - J_{-n}\right).$$

At $(-\infty)$ the argument of the Bessel functions is small and therefore

$$e^{-in\pi} J_n \sim e^{2\pi\kappa h} \left[h \exp\left(\frac{i\alpha}{2} + \frac{x-x_o}{2h}\right) \right]^{2i\kappa h} \Big/ \Gamma(1+2i\kappa h)$$

$$= h^{2i\kappa h} \exp\left[2\kappa h\left(\pi - \frac{\alpha}{2}\right) + i\kappa(x-x_o)\right] \Big/ \Gamma(1+i\kappa h);$$

$$J_{-n} \sim \left[h \exp\left(\frac{i\alpha}{2} + \frac{x-x_o}{2h}\right) \right]^{-2i\kappa h} \Big/ \Gamma(1-2i\kappa h)$$

$$= h^{-2i\kappa h} \exp\left[\kappa h\alpha - i\kappa(x-x_o)\right] \Big/ \Gamma(1-2i\kappa h).$$

The reflection coefficient is the ratio of the complex amplitudes of $\exp\left[-i\kappa(x-x_o)\right]$ and $\exp\left[i\kappa(x-x_o)\right]$. Hence

$$(47) \qquad |R| = \exp\left[-2\kappa h\left(\pi - \frac{\alpha}{2}\right) + \kappa h\alpha\right] = \exp\left[-2\kappa h(\pi - \alpha)\right].$$

In the case of propagation in Region II we have the special cases $\alpha_+ = \pi$ and $\alpha_- = 0$. From (40) we have

$$(48) \qquad \kappa^2_{\pm} = 1 + \frac{1}{2}\tan^2\theta \pm \frac{1}{2}\tan^2\theta.$$

The reflection coefficients are then

$$(49) \qquad |R_+| = 1$$

$$|R_-| = \exp(-2\pi\kappa h \sec\theta).$$

Thus we see that the Π_+ wave is unaffected in Region I and totally reflected in Region II, while the Π_- wave is partially reflected in both regions. For reasonable values of parameters, the partial reflections of the Π_- wave are small. Thus we would expect a single linearly polarized component in the downcoming wave. It is well known that at very low frequencies a single component is observed, but it is approximately circularly polarized. Hence we have to question the validity of the assumptions made. The most vulnerable is the one concerning the smallness of collision frequency. This assumption is not satisfied in those regions of the ionosphere in which the reflection of low frequency waves takes place.

4. Oblique Propagation in Vertical Magnetic Field; Collision Frequency Large

We shall now investigate the case of large collision frequency, $Z \gg Y$. In this case, in Region I we can neglect $X/(1+iZ \pm Y)$ in equation (1) and, in addition, set $m = \tan^2\theta$, since $X \ll Z$. Thus we obtain

$$\frac{d^2}{dz^2} E_x + k^2\cos^2\theta E_x = 0 \tag{50}$$

$$\frac{d^2}{dz^2} E_y + k^2\cos^2\theta E_y = 0, \tag{51}$$

so that in this region propagation is not affected by the presence of the ionosphere. It is only when X becomes comparable to Z that we get any effect at all. In this case we get

$$\frac{d}{dz}\left[(1+m)\frac{d}{dz}E_x\right] + k^2\left(1+i\frac{X}{Z}\right)E_x = 0 \tag{52}$$

$$\frac{d^2}{dz^2} E_y + k^2\left(1+i\frac{X}{Z}\right)E_y = 0. \tag{53}$$

In the limiting case of normal incidence these equations describe the propagation of electromagnetic waves in a medium having the dielectric constant ε:

$$\varepsilon = 1 + iX/Z = 1 + i\omega_o^2/\omega\nu \tag{54}$$

Thus we find that in this approximation the medium becomes isotropic.

5. Vertical Propagation in Oblique Magnetic Field; Collision Frequency Large

Just as in the case of low collision frequency, the coupling between the two characteristic waves Π_+ and Π_- is small when $\nu >> \omega_H$. The dielectric constants, to the same degree of approximation as used in Sec. 3.3, are

$$\varepsilon_{\pm} \approx 1 + iX/Z = 1 + i\omega_0^2/\omega\nu \,, \tag{55}$$

or identical with (54). From (12) and (13) it is readily seen that the characteristictic polarizations are circular, with opposite sense of rotation.

Again the medium proves to be isotropic in this approximation. Yet we know from experimental data that only one of the two circularly polarized waves is reflected. In order to obtain this effect from the theory it is necessary to retain more terms in the dielectric constant expression. We have, from (19),

$$\varepsilon_{\pm} \approx 1 \frac{X}{1+iZ \pm Y\cos\theta} \,, \tag{56}$$

the well-known longitudinal propagation approximation. If X varies as $\exp(z-z_0)/h$, the reflection coefficients are

$$R_{\pm} = \exp(-2kh \tan^{-1} \frac{Z}{1 \pm Y\cos\theta}) \,, \tag{57}$$

where the arctangent is restricted to the range 0 to π. For the case $Z >> Y$, we have

$$\tan^{-1} \frac{Z}{1 \pm Y\cos\theta} \approx \frac{\pi}{2} - \frac{1 \pm Y\cos\theta}{Z} \tag{58}$$

The ratio of the reflection coefficients is then

$$\frac{R_+}{R_-} \approx \exp(-4kh \frac{Y\cos\theta}{Z}). \tag{59}$$

Thus we see that if the increase in electron density is very gradual, i.e., when kh is large, the ratio of the two intensities is very small.

6. Conclusions

As was pointed out in Sec. 3, for reasonable values of parameters we would expect a single downcoming linearly polarized component. Specifically, for the same values of parameters as used by Heading and Whipple, with the electron density varying as exp (z/0.6), the ratio of the two components is less than 0.2 at 16 kc and less than 0.01 at 80 kc. Since at

these frequencies the downcoming wave usually is circularly polarized, the theory apparently does not apply. It is hard to believe that this disagreement could be due to the difference between the actual electron distribution and the assumed exponential one. It is more probable that the neglect of coupling and the assumption of low collision frequency are responsible. Neglecting coupling is particularly dangerous. The example in the Appendix shows that neglecting coupling in a particularly simple case can lead to completely wrong results with respect to transmission and reflection coefficients. Although the example used is a rather special case, it is by no means unreasonable, and the equations treated there bear great resemblance to Rydbeck's equations of propagation in the ionosphere. The agreement between the experimental data and the high collision frequency results is better, but even in this case higher-order approximations have to be used to explain even the most fundamental characteristics of propagation. It seems evident that in order to explain fully the phenomena of low frequency propagation, the theory must not assume the collision frequency to be large or small. However, if these assumptions are dropped the complexity of the equations makes it doubtful whether the problem can be handled analytically. A numerical method seems indicated.

Acknowledgment

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Appendix

Let us consider the system of coupled wave equations

(60) $$u^{\pm}_{xx} + (k^2 - \omega^2)\, u^{\pm} = 2i\omega u^{\mp}_x + i\omega_x\, u^{\mp},$$

where ω is a function of x but k is a constant. This system of equations is identical with Rydbeck's equations of propagation in the ionosphere (cf. Eq. (10)), if we let $\varepsilon_+ = \varepsilon_- = 1$ in the latter and identify ω with $-(1/2)(d\delta/dz)(1+\delta^2)^{-1/2}$. In this simple case we can solve the equations exactly. We define a new set of dependent variables $\nu^{\pm}$

(61) $$\begin{pmatrix} \nu^+ \\ \nu^- \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix},$$

where

(62) $$\theta = \int_{-\infty}^{x} \omega(\xi)\, d\xi \quad ,$$

and it is assumed that the integral exists for all values of x, and that it tends to a constant, θ_o, as $x \to \infty$. Eq. (60) then transforms into

(63) $$\nu^{\pm}_{xx} + k^2\, \nu^{\pm} = 0.$$

Let us take a particular solution of this system, say

(64) $$\nu^+ = e^{ikx}$$ $$\nu^- = 0.$$

Inverting (61), we have

(65) $$\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu^+ \\ \nu^- \end{pmatrix}.$$

Thus we see that the asymptotic forms of this particular solution are

(66) $$x \to -\infty, \quad u^+ \to e^{ikx}; \qquad x \to +\infty, \quad u^+ \to \cos\theta_o\, e^{ikx}$$ $$u^- \to 0 \quad ; \qquad\qquad u^- \to -i\sin\theta_o\, e^{ikx}.$$

Hence, so long as θ_o is not an integral multiple of 2π, the coupling terms cannot be neglected. If we did neglect them, then the asymptotic form at

$(-\infty)$ would be identical with the asymptotic form at $(+\infty)$. Furthermore we can make the coupling coefficients arbitrarily small, and yet, if we keep θ_0 constant, the coupling terms never become negligible. For example, let

$$\begin{aligned} \theta &= \theta_0\, e^{\varepsilon x}\,(e^{-\varepsilon x} + e^{\varepsilon x})^{-1} \\ \omega &= 2a\,\theta_0 (e^{-\varepsilon x} + e^{\varepsilon x})^{-2} \, . \end{aligned} \qquad (67)$$

In this case, although $\theta(\infty) = \theta_0$ is independent of ε, the extrema of ω and ω_x tend to zero as $\varepsilon \to 0$. Hence we see that it makes a difference at what stage ϵ is allowed to approach zero: if we solve Eqs. (60) and then let $\epsilon \to 0$ we get one result, and if we let $\epsilon \to 0$ in Eqs. (60) and now solve them, we get another.

The above example shows clearly that the omission of coupling terms, however small the coefficients may be, is not always justified.

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Errata

Page 11	Eq. (3.2)	In denominator of integrand change " $\cos(\pi/n - \cos\|(\beta+\phi)/\gamma\|$ " to read " $\cos(\pi/x) - \cos[(\beta+\phi)/\gamma]$ "
Page 13,	5th line from bottom	In expression for $\|h_1(\theta,\frac{\pi}{2})\|$, change "$E_1^p$" to read "$H_1^p$"
Page 15,	8th line from bottom	Change " pages (16) and (17)" to read "pages (17-a) and (16) respectively"
Page 15,	4th line from bottom	Omit "for these angles"
Page 15,	4th line from bottom	Change "$h(\pi, \frac{\pi}{2})$" to read "$h_1(\pi, \frac{\pi}{2})$"
Page 17,	Change " $\|e(\theta, \frac{\pi}{2})\|$" to read " $\|e_1(\theta, \frac{\pi}{2})\|$" in caption and in designation of vertical ordinate.	

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